

Janay 2008

PHYS 303.

Solutions.

1 a.

| | | |
|----|-------|-------|
| 5€ | _____ | n_5 |
| 4€ | _____ | n_4 |
| 3€ | _____ | n_3 |
| 2€ | _____ | n_2 |
| € | _____ | n_1 |
| 0 | _____ | n_0 |

$$N = 4$$

$$U = 5€.$$

Distributions

| | n_0 | n_1 | n_2 | n_3 | n_4 | n_5 | Distinguishable Microstates | Indistinguishable Microstates |
|---|-------|-------|-------|-------|-------|-------|--------------------------------|----------------------------------|
| 1 | 3 | 0 | 0 | 0 | 0 | 1 | 4 | 1 |
| 2 | 2 | 1 | 0 | 0 | 1 | 0 | 12 | 1 |
| 3 | 2 | 0 | 1 | 1 | 0 | 0 | 12 | 1 |
| 4 | 1 | 2 | 0 | 1 | 0 | 0 | 12 | 1 |
| 5 | 1 | 1 | 2 | 0 | 0 | 0 | 12 | 1 |
| 6 | 0 | 3 | 1 | 0 | 0 | 0 | 4 | 1 |
| | | 2 | | | | | <u>56</u> | <u>6</u> |
| | | | | | | | $\Omega = 56$ | $\Omega = 6$ |

Distinguishable

Evaluation of populations

$$\bar{n}_0 = \frac{1}{56} \{ 4 \times 2 + 12 \times 2 + 12 \times 2 + 12 \times 1 + 12 \times 1 \} = \frac{84}{56} = 1.50$$

$$\bar{n}_1 = \frac{1}{56} \{ 12 \times 1 + 12 \times 2 + 12 \times 1 + 4 \times 3 \} = \frac{60}{56} = 1.07$$

$$\bar{n}_2 = \frac{1}{56} \{ 12 \times 1 + 12 \times 2 + 4 \times 1 \} = \frac{40}{56} = 0.71$$

$$\bar{n}_3 = \frac{24}{56} = 0.43$$

$$\bar{n}_4 = \frac{12}{56} = 0.21$$

$$\bar{n}_5 = \frac{4}{56} = 0.07$$

Indistinguishable populations

$$\bar{n}_0 = \frac{1}{6} \{ 3 + 2 + 2 + 1 + 1 \} = \frac{9}{6} = 1.5$$

$$\bar{n}_1 = \frac{7}{6}$$

$$\bar{n}_2 = \frac{4}{6}$$

$$\bar{n}_3 = \frac{2}{6}$$

$$\bar{n}_4 = \frac{1}{6}$$

$$\bar{n}_5 = \frac{1}{6}$$

1 b.

$$\frac{3\epsilon}{2} \text{ —————}$$

N atoms.

Energy levels $\epsilon/2$, $3\epsilon/2$

Temperature T .

$$\epsilon/2 \text{ —————}$$

(i) Partition function $Z = \exp(-\epsilon/2kT) + \exp(-3\epsilon/2kT)$

$$= \exp(-\epsilon/2kT) [1 + \exp(-\epsilon/kT)] \quad 2$$

(ii) $U = NkT^2 \frac{\partial}{\partial T} \ln \left\{ \exp(-\epsilon/2kT) [1 + \exp(-\epsilon/kT)] \right\}$

$$= NkT^2 \frac{\partial}{\partial T} \left\{ (-\epsilon/2kT) + \ln [1 + \exp(-\epsilon/kT)] \right\}$$

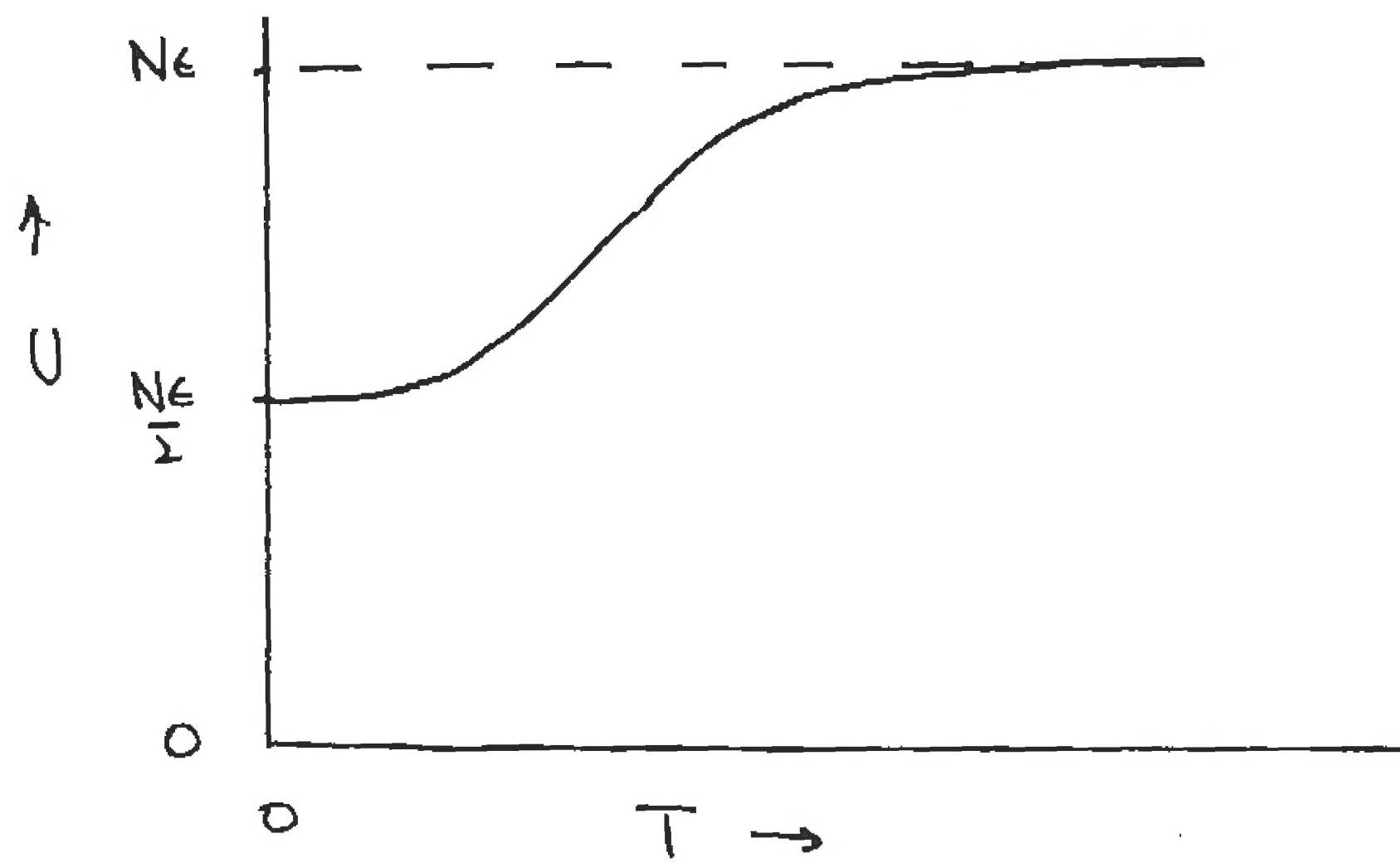
$$= NkT^2 \cdot \left(\frac{\epsilon}{2kT^2} \right) + \frac{NkT^2 \cdot (\epsilon/kT^2) \exp(-\epsilon/kT)}{[1 + \exp(-\epsilon/kT)]}$$

$$U = \frac{N\epsilon}{2} + \frac{N\epsilon \exp(-\epsilon/kT)}{[1 + \exp(-\epsilon/kT)]} \quad 2$$

(iii) As $T \rightarrow 0$ $\exp(-\epsilon/kT) \rightarrow 0$ $U \rightarrow \frac{N\epsilon}{2} \quad 1$

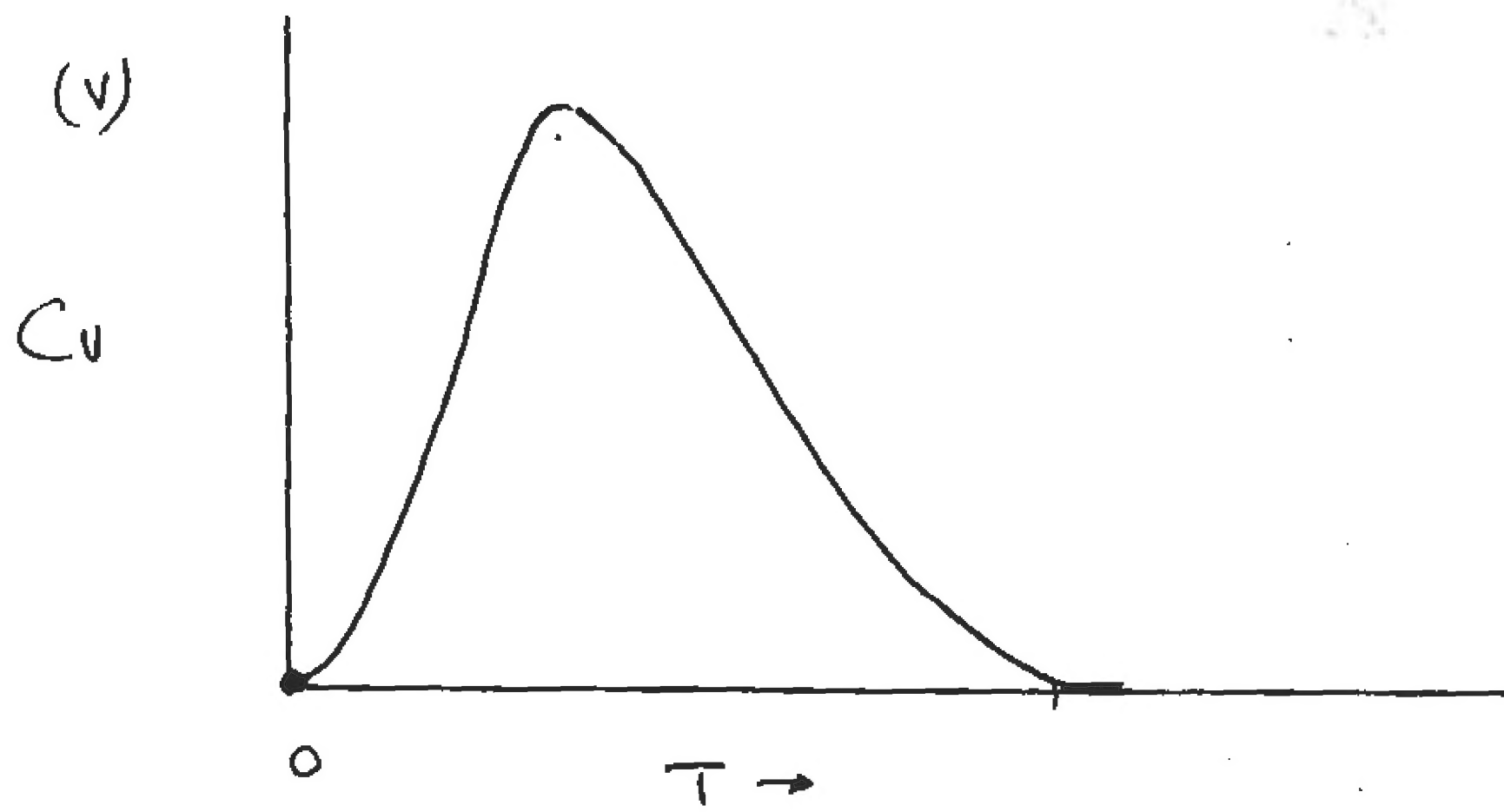
As $T \rightarrow \infty$ $\exp(-\epsilon/kT) \rightarrow 1$ $U \rightarrow \frac{N\epsilon}{2} + \frac{N\epsilon}{2} = N\epsilon \quad 1$

(iv)



2

(v)



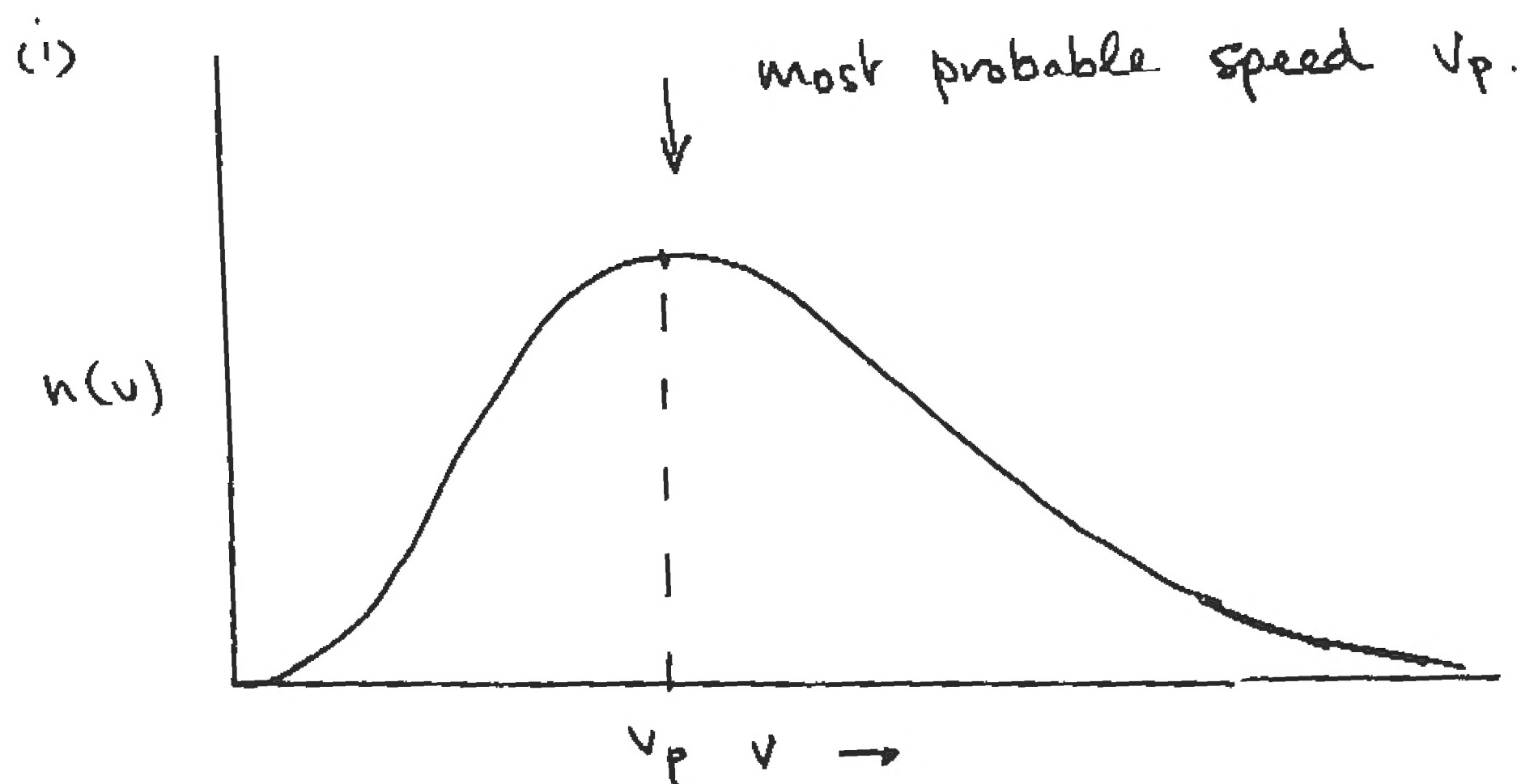
$$C_v = \left(\frac{\partial U}{\partial T} \right)_v$$

is slope of U vs T graph.

1

1c.

Maxwell - Boltzmann distribution



(ii) Max at $\frac{dn}{dv} = 0 = 2Cv \cdot \exp\left(-\frac{mv^2}{2kT}\right) + Cv^2 \cdot \left(\frac{-2mv}{2kT}\right) \exp\left(-\frac{mv^2}{2kT}\right)$

Thus $2\cancel{v} \exp\left(-\cancel{\frac{mv^2}{2kT}}\right) = \cancel{v^2} \left(\frac{mv}{kT}\right) \exp\left(-\cancel{\frac{mv^2}{2kT}}\right)$

$$2\cancel{v} = \cancel{v^2} \cdot \frac{mv}{kT} \quad v^2 = \frac{2kT}{m}$$

$v_p = \left(\frac{2kT}{m}\right)^{1/2}$ — most probable speed. 2

(iii) Mean square speed v_m^2

$b = \left(\frac{m}{2kT}\right)$

$$\overline{v_m^2} = \frac{\int_0^\infty v^2 \cdot Cv^2 \exp\left(-\frac{mv^2}{2kT}\right) dv}{\int_0^\infty Cv^2 \exp\left(-\frac{mv^2}{2kT}\right) dv}$$

$$= \frac{\overline{I}_4}{\overline{I}_2} = \frac{3}{2b} \cdot \frac{\overline{I}_2}{\overline{I}_2}$$

$$\overline{v_m^2} = \frac{3}{2} \cdot \frac{kT}{m} = \frac{3kT}{m}$$

2

(vi) Mean energy of monatomic molecule $= \bar{\epsilon} = \frac{1}{2} m \bar{v}_m^2$

$$\bar{\epsilon} = \frac{1}{2} \cdot \cancel{m} \cdot \frac{3kT}{\cancel{m}} = \frac{3}{2} kT$$

At 300K $\bar{\epsilon} = 1.5 \times 1.38 \times 10^{-23} \times 300 = 6.21 \times 10^{-21} \text{ J.}$ 1

(v) For diatomic molecule at 300K

Translational energy $= \frac{3}{2} kT = 6.21 \times 10^{-21} \text{ J}$

+ Rotational energy $= kT = 4.16 \times 10^{-21} \text{ J.}$

— No vibrational energy.

Thus total energy $= 10.37 \times 10^{-21} \text{ J}$

$= 1.04 \times 10^{-20} \text{ J.}$ 2

1d.

(i) For fermions.

$$f(\epsilon) = \frac{1}{\beta \exp(\epsilon/kT) + 1} \quad \approx \frac{1}{\exp\left(\frac{\epsilon - \epsilon_F}{kT}\right) + 1}$$

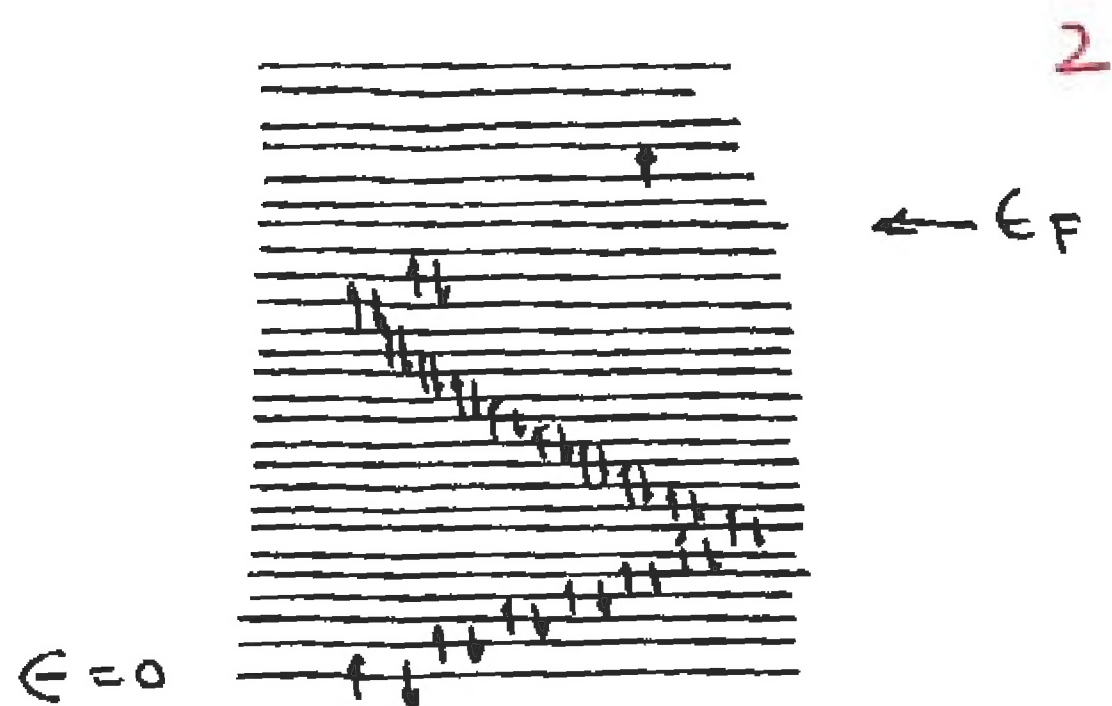
(ii) For bosons

$$f(\epsilon) = \frac{1}{\beta \exp(\epsilon/kT) - 1}$$

(iii) Difference arises because fermions obey Pauli principle which forbids multiple occupancy of same quantum state. Bosons do not obey Pauli principle

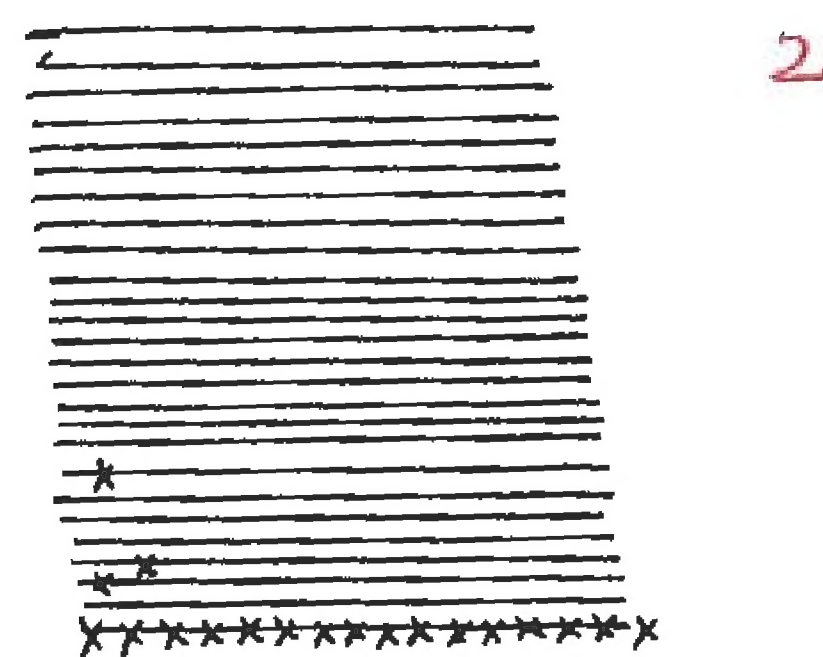
As $T \rightarrow 0$

For fermions



Fermions fill levels $\epsilon < \epsilon_F$ with $\uparrow\downarrow$
Very ~~few~~ small
occupation $\epsilon > \epsilon_F$

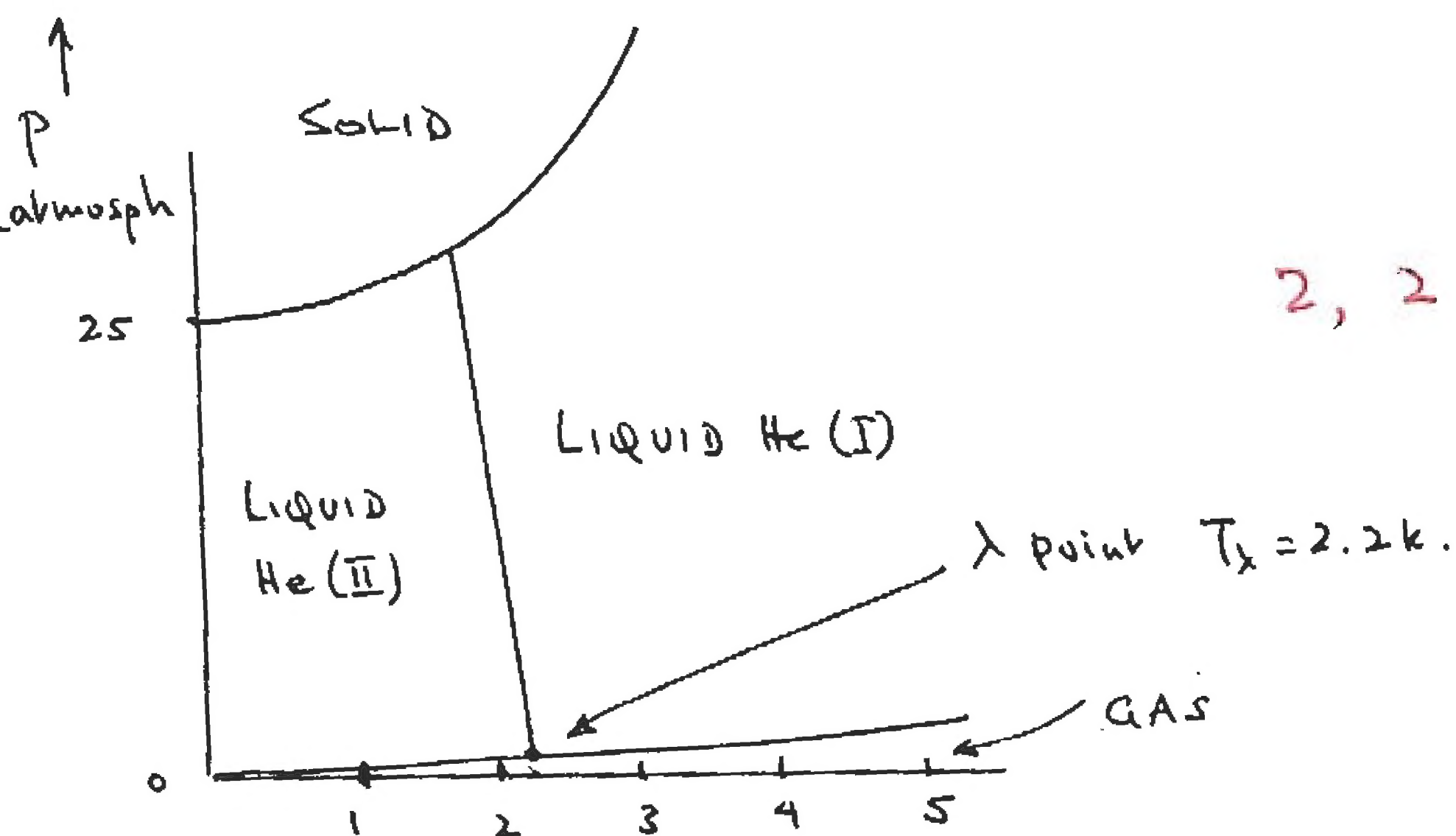
For boson



Bosons as $T \rightarrow 0$
condense into g/s state
heavy occupation of g/s state
very small occupation of
excited states

1.e. Phase diagram of He^4 .

(i), (ii)



(iii). Viscosity anomalous in liquid $\text{He}(\text{II})$

Value Depends on method of measurement.

Method I. — Moment of inertia of set of discs doing torsional oscillation in bath of liquid $\text{He}(\text{II})$

Gives finite viscosity value — viscosity decreases as T decreases

Method II. — Flow thru capillary — gives zero value for viscosity of liquid $\text{He}(\text{II})$.

(iv) Explanation. Liquid $\text{He}(\text{II})$ behaves as if it were composed of 2 intermixed fluids. Normal fluid (with viscosity) and Superfluid (without viscosity)

Flow method sees Superfluid component — zero viscosity

Oscillating discs method sees normal fluid sticking to discs —

finite value. As $T \rightarrow 0$ proportion of normal fluid decreases — viscosity

1 f. Superconductor is a material in which - below a critical temperature T_c electrical current flows with no electrical resistance. 2

BCS theory.

In BCS theory the electric current is carried by electron pairs called Cooper pairs. In these pairs spins are paired ($\uparrow\downarrow$) and $l=0$. 2

These Cooper pairs are bosons and below a critical temperature T_c they undergo a Bose condensation into the ground state to form a condensate which gives the superconductivity flow. 2

The pairing mechanism of 2 electrons \rightarrow Cooper pair occurs by an ~~electron~~ electron-lattice-electron interaction.

The strength of this interaction determines the critical temperature.

Pair breaking by temperature or by applied magnetic field destroys the superconductivity.

Scattering which does not break up pairs has no effect of resistance 2

2a.

(i) Conditions for quantisation.

Standing wave

$$\sin(k_x L) = 0$$

$$k_x L = \pi n_x$$

$$k_y L = \pi n_y$$

$$k_z L = \pi n_z$$

n_x, n_y, n_z are integers

or

Periodic

$$\exp(i k_x L) = 1$$

$$k_x L = 2\pi n_x$$

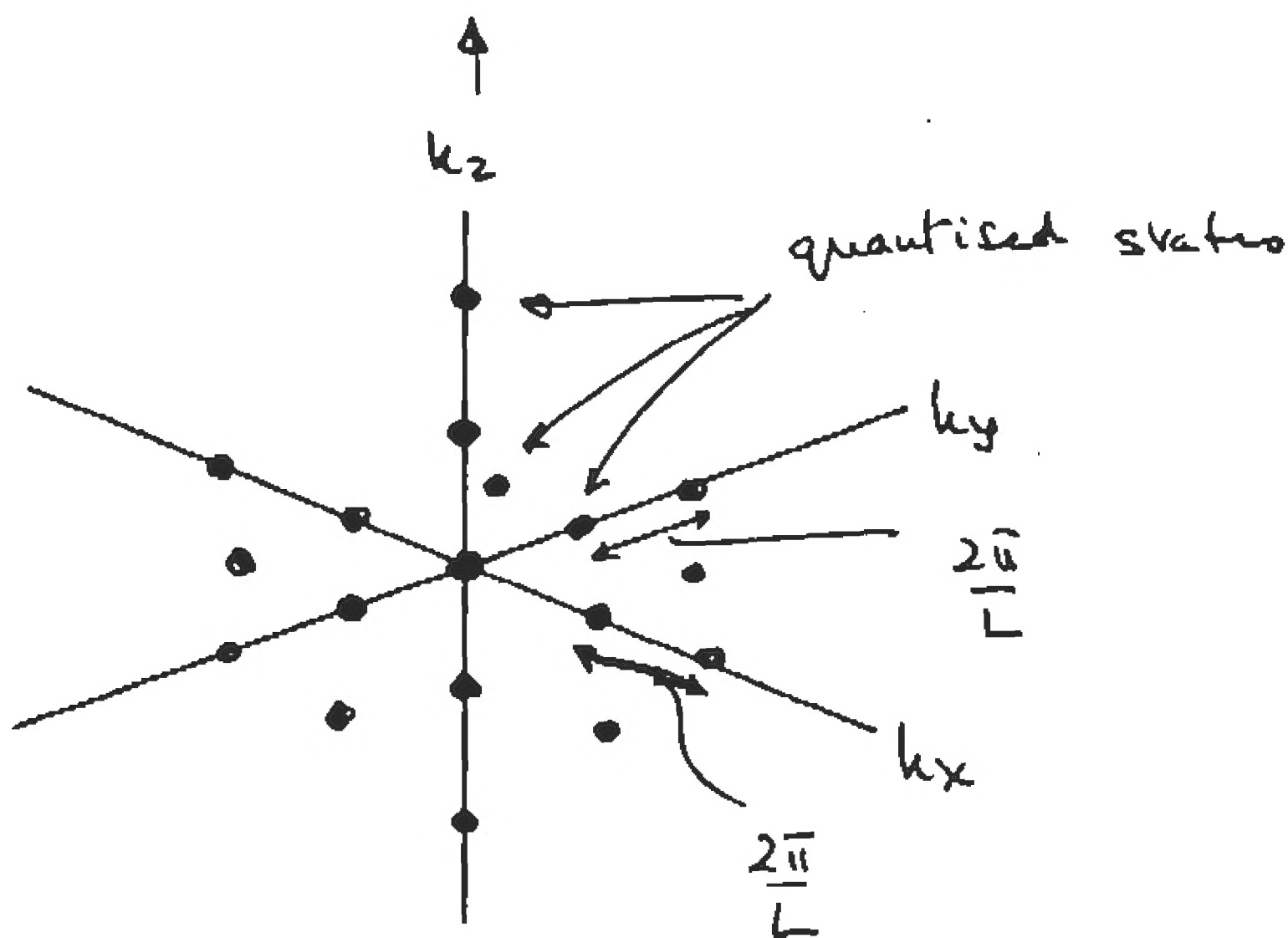
$$k_y L = 2\pi n_y$$

$$k_z L = 2\pi n_z$$

n_x, n_y, n_z integers (+ or -)

2

(ii) Periodic case



3

(iii) $g(k) dk = 2 \times \frac{\text{Volume of space between } k \text{ and } k+dk}{\text{Volume / state} = \left(\frac{2\pi}{L}\right)^3}$

electron spin $\uparrow \downarrow$

$$= 2 \times \frac{4\pi k^2 dk \cdot L^3}{(2\pi)^3}$$

$$= \frac{2 \cdot V \times 4\pi k^2 dk}{(2\pi)^3}$$

where $V = L^3$.

3

2a (contd)

$$(iv) \quad \epsilon = \frac{\hbar^2 k^2}{2m} \quad \text{since } \text{momentum } p = \hbar k.$$

$$(v) \quad g(k) dk = \frac{2 \cdot V \cdot 4\pi k^2 dk}{(2\pi)^3}$$

$$\text{but } k^2 = \frac{2m\epsilon}{\hbar^2} \quad k = \left(\frac{2m\epsilon}{\hbar^2}\right)^{1/2} \quad dk = \left(\frac{2m}{\hbar^2}\right)^{1/2} \cdot \frac{\epsilon^{-1/2}}{2} d\epsilon$$

$$\text{Thus } g(\epsilon) d\epsilon = \frac{2 \cdot V \cdot 4\pi}{(2\pi)^3} \cdot \left(\frac{2m\epsilon}{\hbar^2}\right) \cdot \left(\frac{2m}{\hbar^2}\right)^{1/2} \cdot \frac{1}{2\epsilon^{1/2}} d\epsilon$$

$$g(\epsilon) d\epsilon = \frac{2V}{(2\pi)^2} \cdot \left(\frac{2m}{\hbar^2}\right)^{3/2} \cdot \epsilon^{1/2} d\epsilon$$

$$g(\epsilon) d\epsilon = \frac{2V}{(2\pi)^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \epsilon^{1/2} d\epsilon$$

(vi) Fermi energy.

At $T=0$ all lowest energy states populated up to Fermi sphere in k space — all states outside sphere are empty.

Radius of Fermi sphere = k_F

$$\text{No states} = \underset{\substack{\uparrow \\ \text{spin } 1/2}}{2} \cdot \frac{4\pi}{3} k_F^3 \cdot \frac{1}{(2\pi)^3} = N \quad \leftarrow \begin{matrix} \text{no} \\ \text{electrons.} \end{matrix}$$

$$k_F^3 = \frac{3\pi^2 N}{L^3} = \frac{3\pi^2 N}{V}$$

$$\text{Fermi energy } \mu(0) = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m} \left(\frac{3\pi^2 N}{V}\right)^{2/3}$$

(vii). For Cu. fcc - 4 atoms / unit cube

$$\text{Thus } \frac{N}{V} = \frac{4}{(3.61 \times 10^{-10})^3}$$

$$\mu(0) = \frac{\hbar^2}{2m} \left(3\pi^2 \left(\frac{N}{V} \right) \right)^{2/3} = \left(\frac{6.63 \times 10^{-34}}{2\pi} \right)^2 \cdot \frac{1}{2 \times 9.11 \times 10^{-31}} \left(\frac{3\pi^2 \cdot 4}{(3.61 \times 10^{-10})^3} \right)^{2/3} \text{ J}$$

$$\mu(0) = 1.14 \times 10^{-18} \text{ J} \approx \frac{1.14 \times 10^{-18}}{1.60 \times 10^{-19}} = 7.1 \text{ eV. } 3$$

(viii) Electronic heat capacity:

$$C_v = \left(\frac{\partial U}{\partial T} \right)_v \quad U = \frac{3}{5} N \mu(0) + \frac{\pi^2 (kT)^2 \cdot 3N}{12 \mu(0)}$$

$$C_v = \frac{\cancel{3} N \pi^2 k^2 \cdot 2T}{2 \cancel{12} \mu(0)} = \frac{6.02 \times 10^{23} \times \pi^2 \times (1.38 \times 10^{-23})^2 \times 4.2}{2 \times 1.14 \times 10^{-18}} \text{ J/K}$$

$$C_v = 2.08 \times 10^{-3} \text{ J/K. } 3$$

$$(ix) C_v(\text{total}) = C_v(\text{lattice}) + C_v(\text{electrons})$$

$$\begin{aligned} C_v(\text{lattice}) &= 234 \times 6.02 \times 10^{23} \times 1.38 \times 10^{-23} \cdot \left(\frac{4.2}{343} \right)^3 \\ &= 3.57 \times 10^{-3} \text{ J/K} \end{aligned}$$

$$C_v(\text{total})_{\text{molar}} = 3.57 \times 10^{-3} + 2.08 \times 10^{-3} = 5.65 \times 10^{-3} \text{ J/K} 3$$

2(b). Quantised electromagnetic radiation.

(i) $\nu = \frac{c}{\lambda}$, $h = \frac{2\pi}{\lambda}$ (hence $\nu = \frac{c}{2\pi h}$) 2

(ii) $g(k) = \frac{2 \cdot V \cdot 4\pi k^2 dk}{(2\pi)^3}$

$$k^2 = \left(\frac{2\pi\nu}{c}\right)^2 \quad dk = \frac{2\pi}{c} d\nu$$

thus $g(\nu) = \frac{2 \cdot V \cdot 4\pi}{(2\pi)^3} \cdot \left(\frac{2\pi\nu}{c}\right)^2 \left(\frac{2\pi}{c}\right) d\nu$
 $= \frac{8\pi V \cdot \nu^2 d\nu}{c^3}$ 3

(iii) $\mathcal{E}(\nu) d\nu = \text{density of states} \times \text{energy/photon} \times \text{prob photon state occupied}$

$$h\nu = \text{energy / photon}$$
 1

$$\left[\exp(h\nu/kT) - 1 \right]^{-1} = \text{probability photon state occupied}$$
 2

(iv) As $\nu \rightarrow 0$

$$\exp(h\nu/kT) \rightarrow \exp(0) \rightarrow 1 + (h\nu/kT)$$

$$\mathcal{E}(\nu) \rightarrow \frac{8\pi V \nu^2}{c^3} \cdot \cancel{h\nu} \cdot \left(\frac{kT}{h\nu}\right) \rightarrow 0 \quad \left(\text{as } \nu^2\right)$$
 2

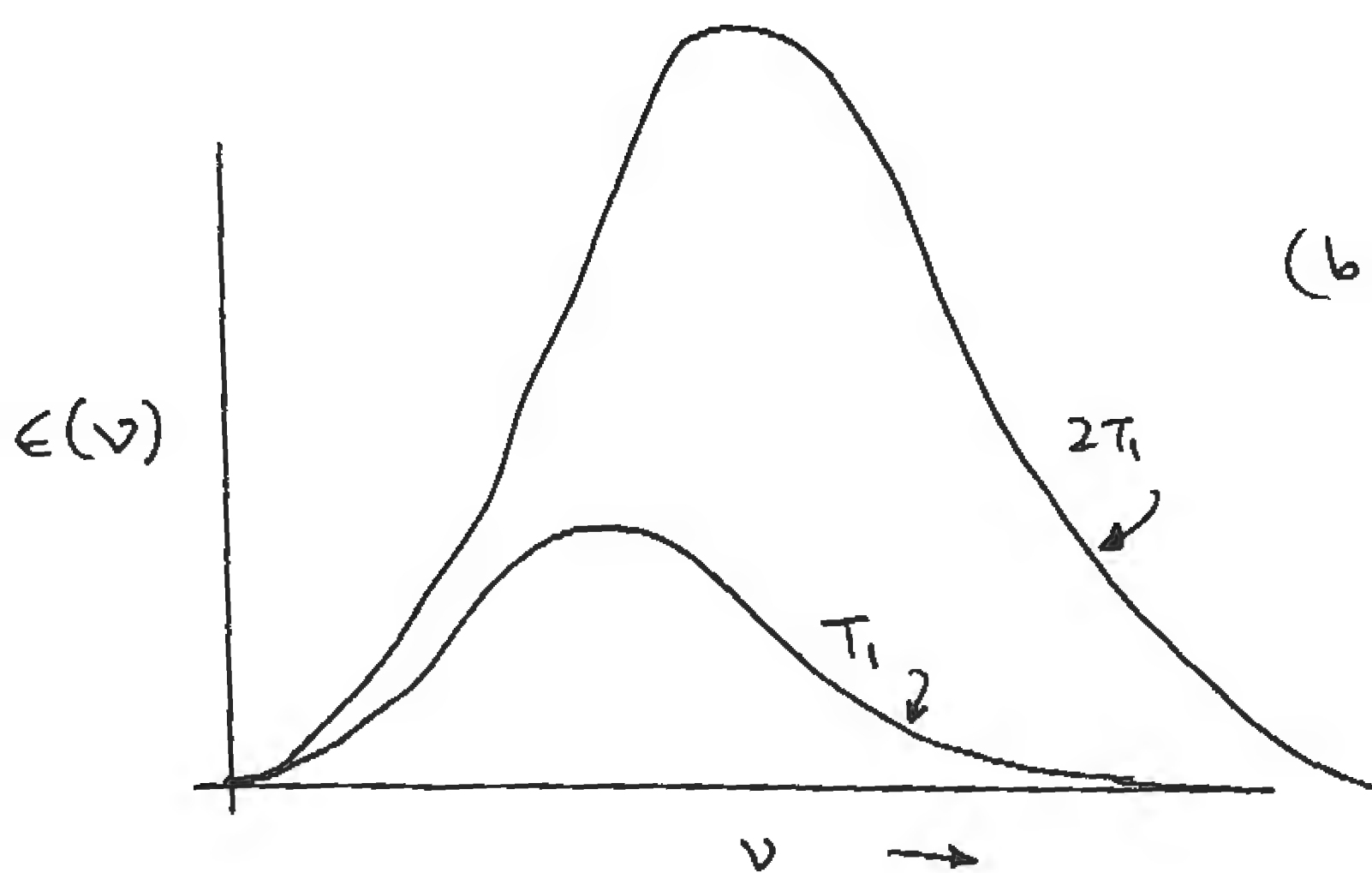
$$\text{As } \nu \rightarrow \infty \quad \exp(h\nu/kT) \rightarrow \infty$$

Denom $\rightarrow \infty$ faster than numerator (ν^2)

Thus $\mathcal{E}(\nu) \rightarrow 0$ as $\nu \rightarrow \infty$

 2

(v)



Points

(a) max for $2T_1$ at larger value of ν than that for T_1

(b) Area under $2T_1$ curve greater than that under T_1 curve

1, 1

$$(vi) \quad \epsilon(\nu) = \frac{8\pi V \nu^2 h \nu}{c^3} \cdot \frac{1}{[\exp(h\nu/kT) - 1]}$$

For max $\epsilon(\nu)$ $\left(\frac{d\epsilon}{d\nu}\right) = 0$

$$\left(\frac{d\epsilon}{d\nu}\right) = \frac{8\pi V h}{c^3} \left\{ \frac{[\exp(h\nu/kT) - 1] \cdot 3\nu^2 - \nu^3 \cdot \left(\frac{h}{kT}\right) \exp(h\nu/kT)}{[\exp(h\nu/kT) - 1]^2} \right\} = 0$$

$$\{ \} = 0$$

thus. $3\nu^2 \exp(h\nu/kT) - 3\nu^2 = \frac{h\nu^3}{kT} \exp(h\nu/kT)$

Put $x = h\nu/kT$.

$$3 \exp(x) - 3 = x \exp(x)$$

$$(3 - x) \exp(x) = 3$$

$$\exp(x) = \frac{3}{3-x}$$

2

Check solution for $x = 2.82$

$$\exp(2.82) = 16.8$$

$$\frac{3}{3-2.82} = \frac{3}{0.18} = 16.7.$$

} OK solⁿ $x = 2.82$.

$$\nu_m = \frac{2.82 kT}{h}$$

(vii) For $T_1 = 6000 \text{ K}$

$$\nu_m = \frac{2.82 \times 1.38 \times 10^{-23} \times 6000}{6.63 \times 10^{-34}} = 3.52 \times 10^{14} \text{ s}^{-1}$$

To recognise get equivalent λ

$$\lambda = \frac{c}{\nu} = \frac{3 \times 10^8}{3.52 \times 10^{14}} = 8.52 \times 10^{-7} \text{ m} = \text{u/violet radiation.} \quad 2$$

For $T_2 = 2.9 \text{ K}$

$$\nu_m = \frac{2.82 \times 1.38 \times 10^{-23} \times 2.9}{6.63 \times 10^{-34}} = 1.60 \times 10^{11} \text{ s}^{-1}$$

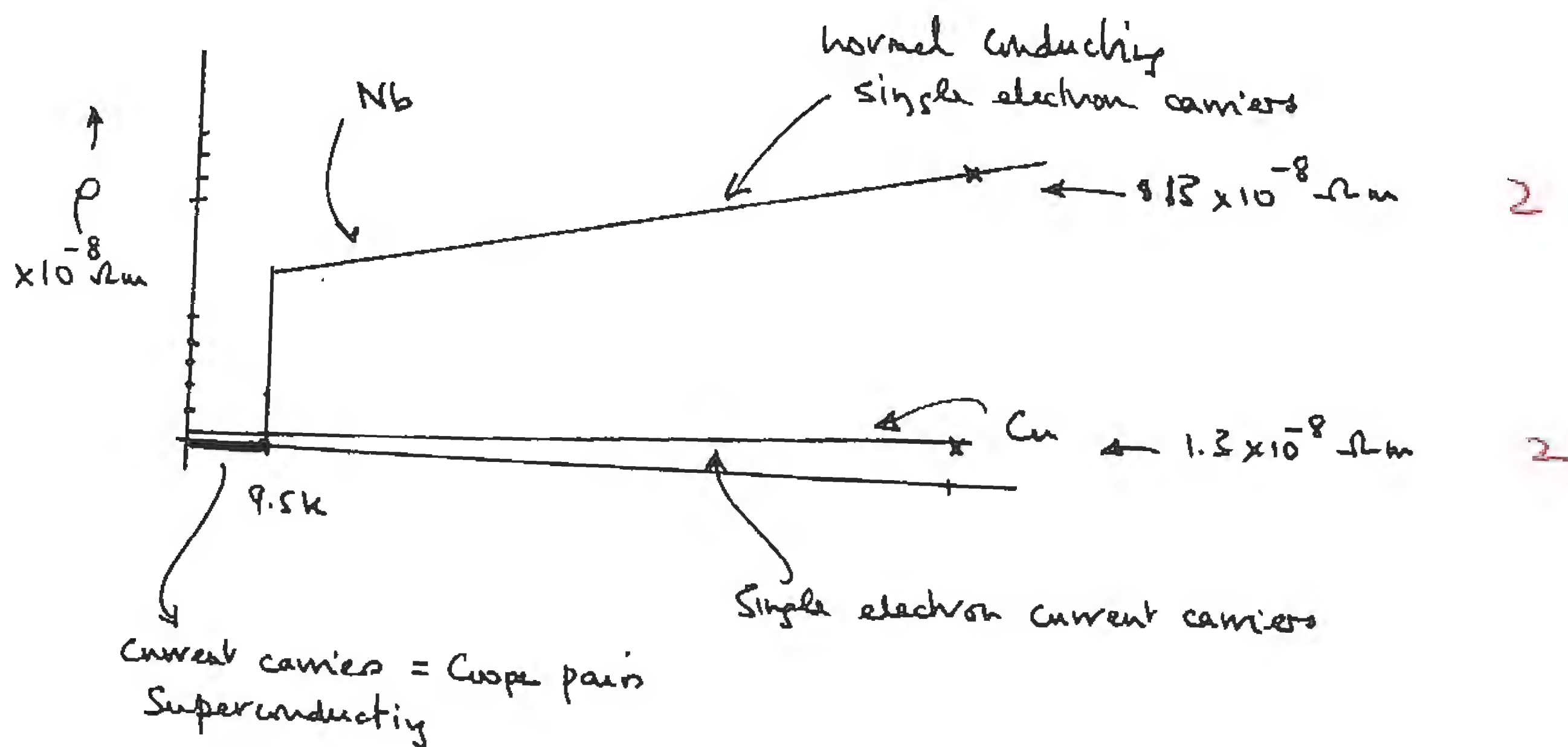
$$\lambda = \frac{c}{\nu} = \frac{3 \times 10^8}{1.60 \times 10^{11}} = 1.87 \times 10^{-3} \text{ m} \quad \text{microwaves} \quad 2$$

(viii) The microwave background — black body radiation spectrum with a characteristic temperature of 2.9 K is predicted from the 'big bang' and subsequent expansion of the Universe. Treating Universe as a box which has expanded — this should be e/m background from big bang event.

3

3(a)

(i) ρ versus T graphs.



Cu — always single electron current carriers

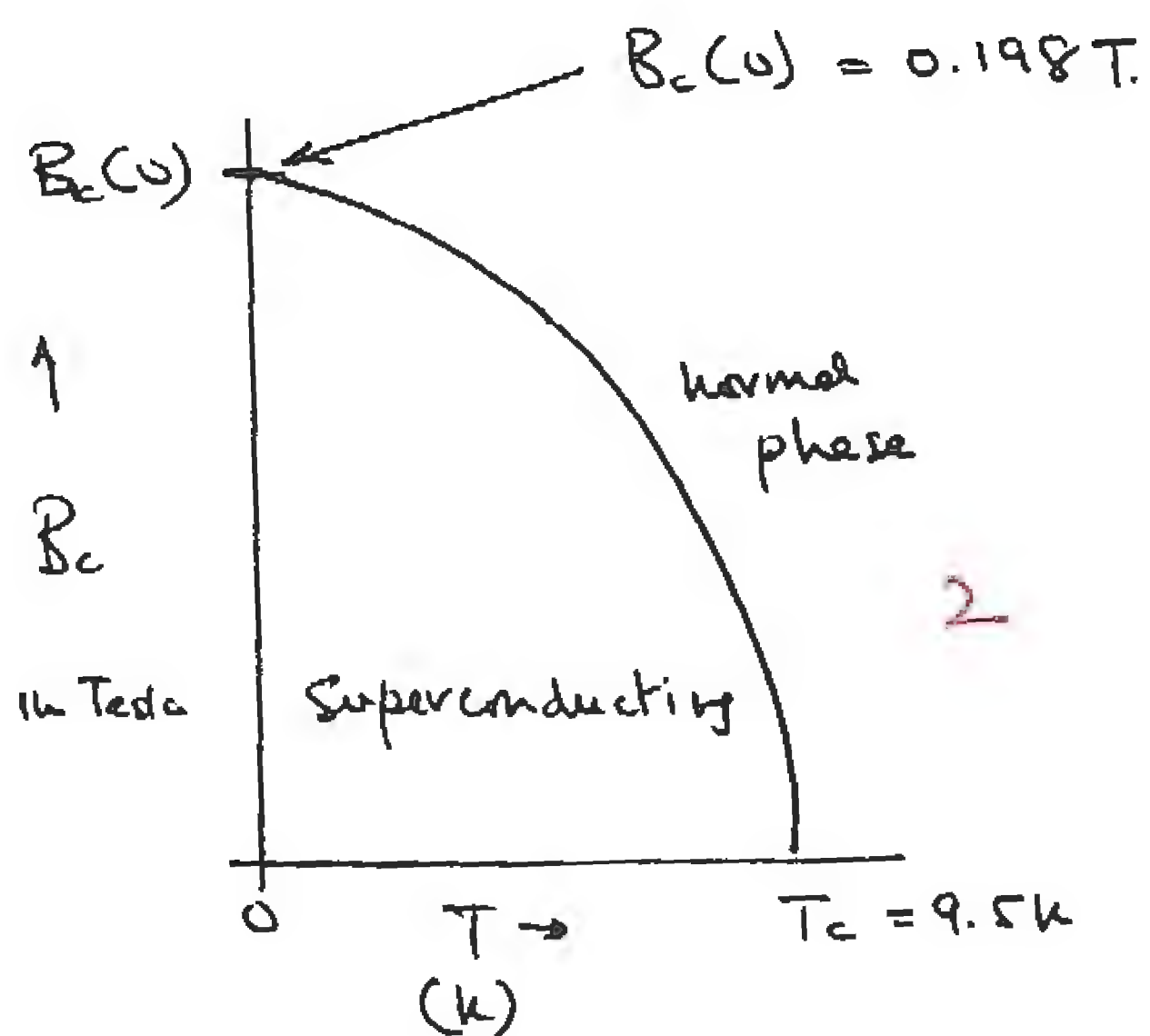
Nb — Superconducting — Cooper pair current carriers
 Normal phase — single electron current carriers.

(ii) For superconducting mechanism — electron pairing via electron-lattice — electron interaction — must have strong electron-lattice interaction. But in normal phase the strong electron lattice interaction results in strong single electron scattering and high resistance.

In Cu — weak electron lattice interaction — too weak for superconducting but in single electron conduction weak interaction leads to low resistance. 3

(iii) Applied field B acts to align electron spins \parallel to B . This breaks $\uparrow\downarrow$ Cooper pairs and destroys $\rho = 0$. 2

(iv)



$$B_c(T) = B_c(0) \left[1 - \left(\frac{T}{T_c} \right)^2 \right]$$

(v) For Nb $T_c = 9.5 \text{ K}$ and $B_c(0) = 0.198 \text{ T}$.

Critical field at 4.2 K

$$B_c(T) = 0.198 \left[1 - \left(\frac{4.2}{9.5} \right)^2 \right] = 0.198 \left[1 - 0.195 \right]$$

$$= 0.198 \times 0.805 = 0.16 \text{ T}$$

Critical temperature in a field of 0.10 T

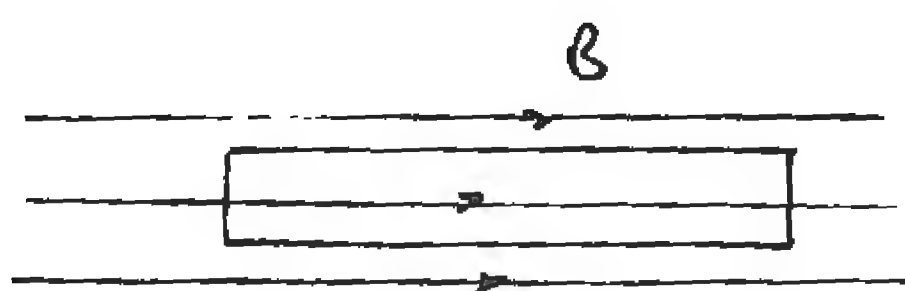
$$0.10 = 0.198 \left[1 - \left(\frac{T}{9.5} \right)^2 \right]$$

$$\frac{0.10}{0.198} = 0.505 = 1 - \left(\frac{T}{9.5} \right)^2$$

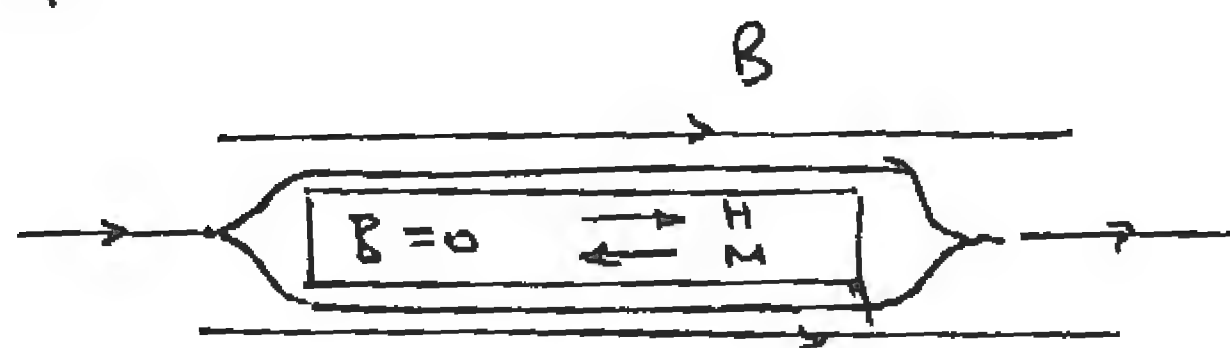
$$\left(\frac{T}{9.5} \right)^2 = 1 - 0.505 = 0.495$$

$$T = 9.5 \times (0.495)^{1/2} = 9.5 \times 0.70 = 6.7 \text{ K}$$

(vi) Meissner - Oschenfeld effect



Normal state



Superconducting state

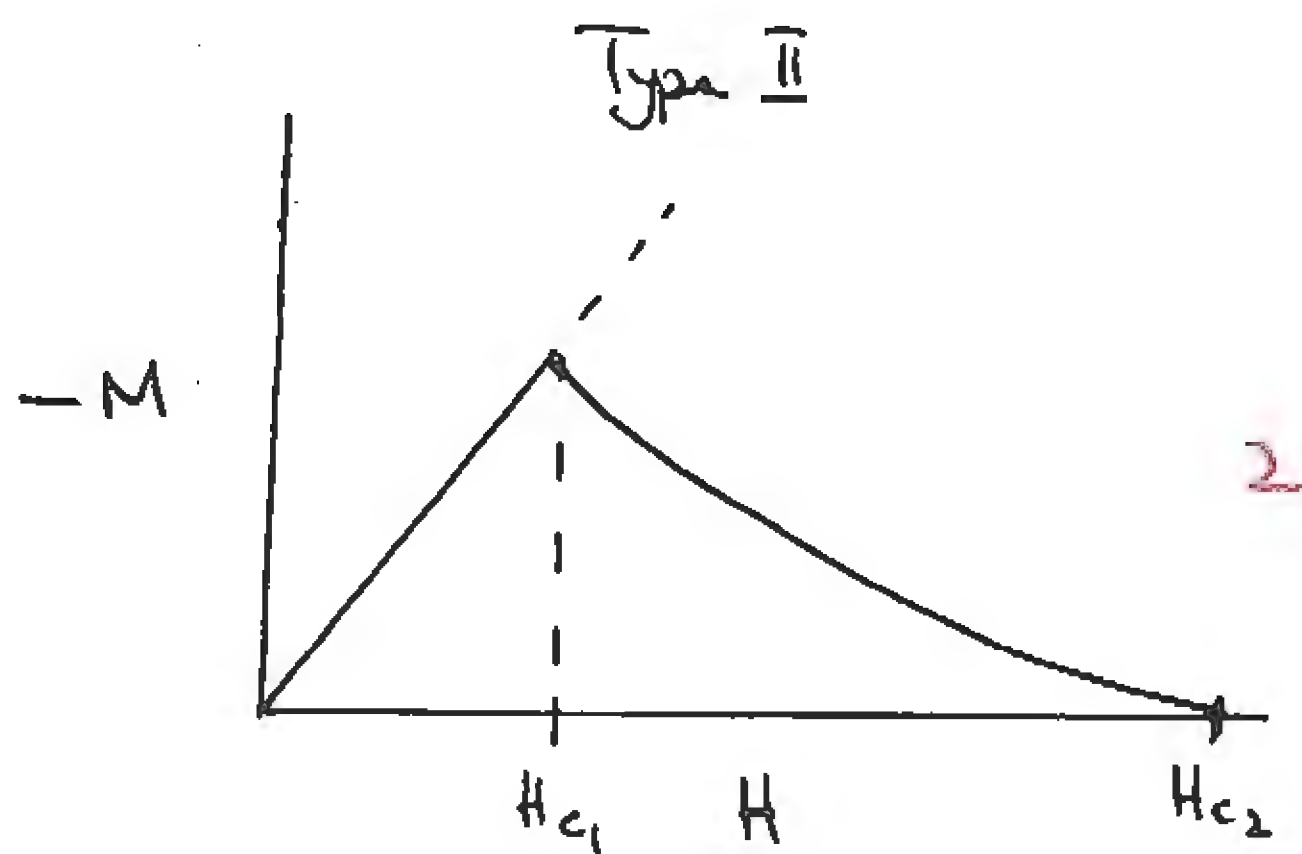
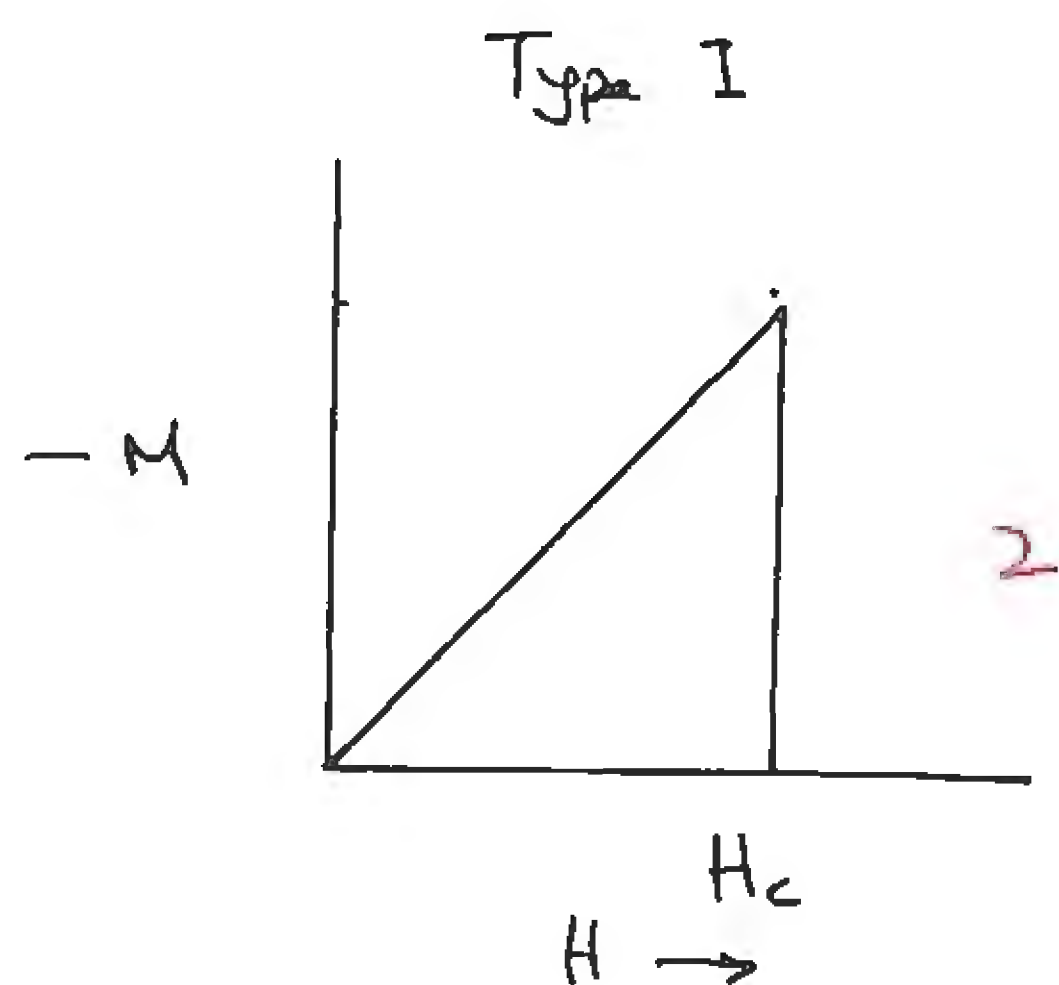
For an applied field $B < B_c$ on a superconducting sample the lines of B are excluded from the bulk of the superconducting sample. In normal state - no expulsion of lines of B .

Expulsion occurs because induced currents on surface of superconductor give rise to a magnetisation ($-M$) that cancels the applied field H and gives rise to $B=0$ via the relation

$$B = \mu_0 (H + M) \quad - \text{ for superconductor } M = -H$$

$$\rightarrow \text{hence } B = \mu_0 (H - H) = 0.$$

(vii)



(viii) Explanation of Type II graph

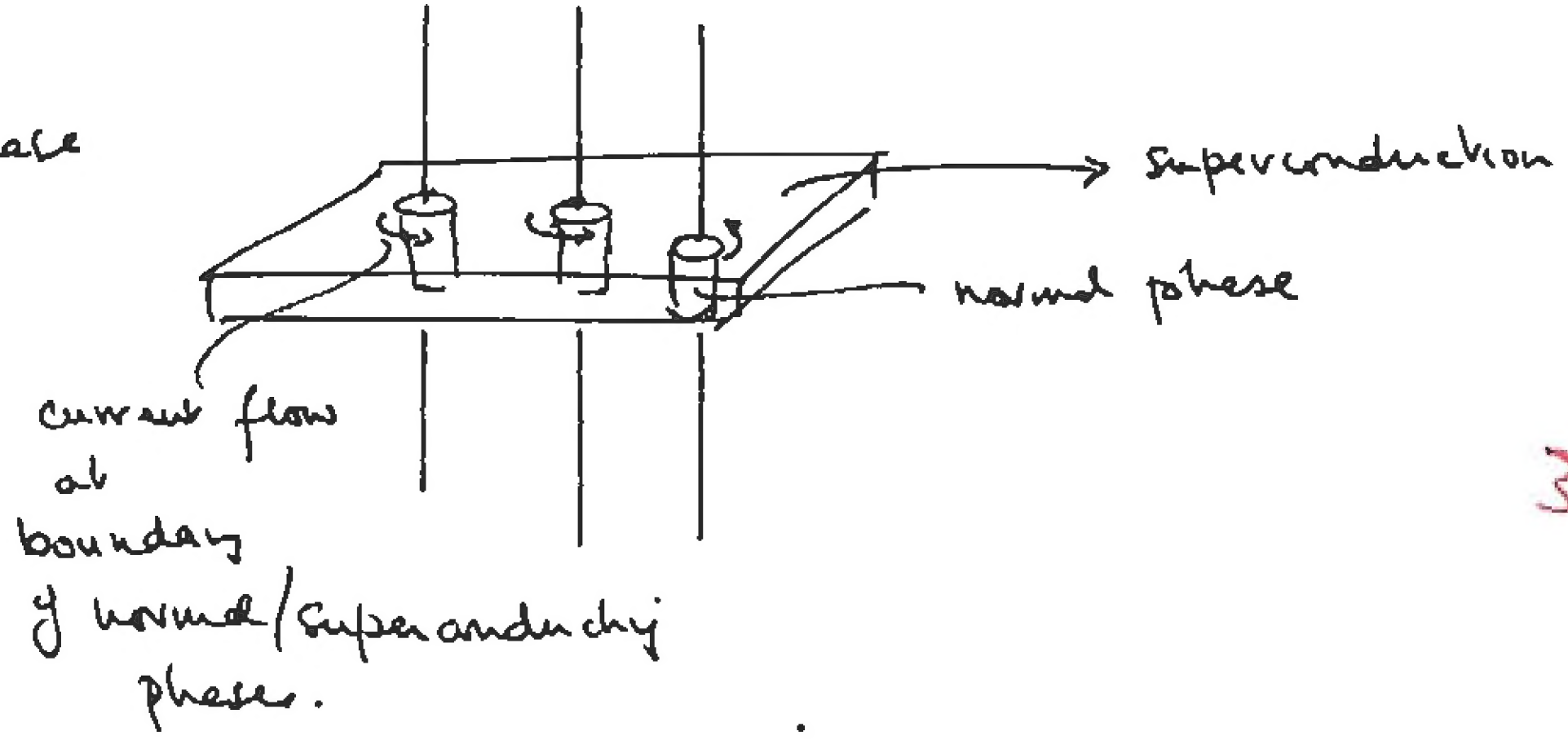
For $0 < H < H_{c1}$ - all flux B excluded

$H_{c1} < H < H_{c2}$ mixed phase - some lines of H penetrate sample. Around each H line is small normal region

- see diagram

$H_{c2} < H$ - sample normal

Mixed phase



3(b)

(i) Superfluid He^3

Logical account — including some of points

He^3 superfluid at $T < 2 \text{ mK}$

Pairing of two He^3 atoms. \rightarrow bosons — condensation to superfluid phase

Pairing interaction weak (low T_c)

Pairs here $L=1$, $S=1$.

Different superfluid phases A, B — contain different

proportions of $S=1$ $S_z = \cancel{+1}, 0, -1$ pairs

Pairs have magnetic moment — phase diagram affected by

applied magnetic field

5

(ii) Identify cooling techniques

(a) to 77 K — liquid N_2 bath at atmospheric pressure

(b) to 4.2 K — liquid He^4 bath at atmospheric pressure

(c) to 2 K — liquid He bath at reduced pressure

(d) to $5 \times 10^{-3} \text{ K}$ — dilution refrigerator or Pomaranchuk cell

(e) to $5 \times 10^{-6} \text{ K}$ — nuclear adiabatic demagnetisation

5

(iii) Pomaranchuk cooling. - logical account including some of points

Cooling medium He^3

In temp range $2mK \rightarrow 0.33K$ entropy of liquid $<$ entropy of solid

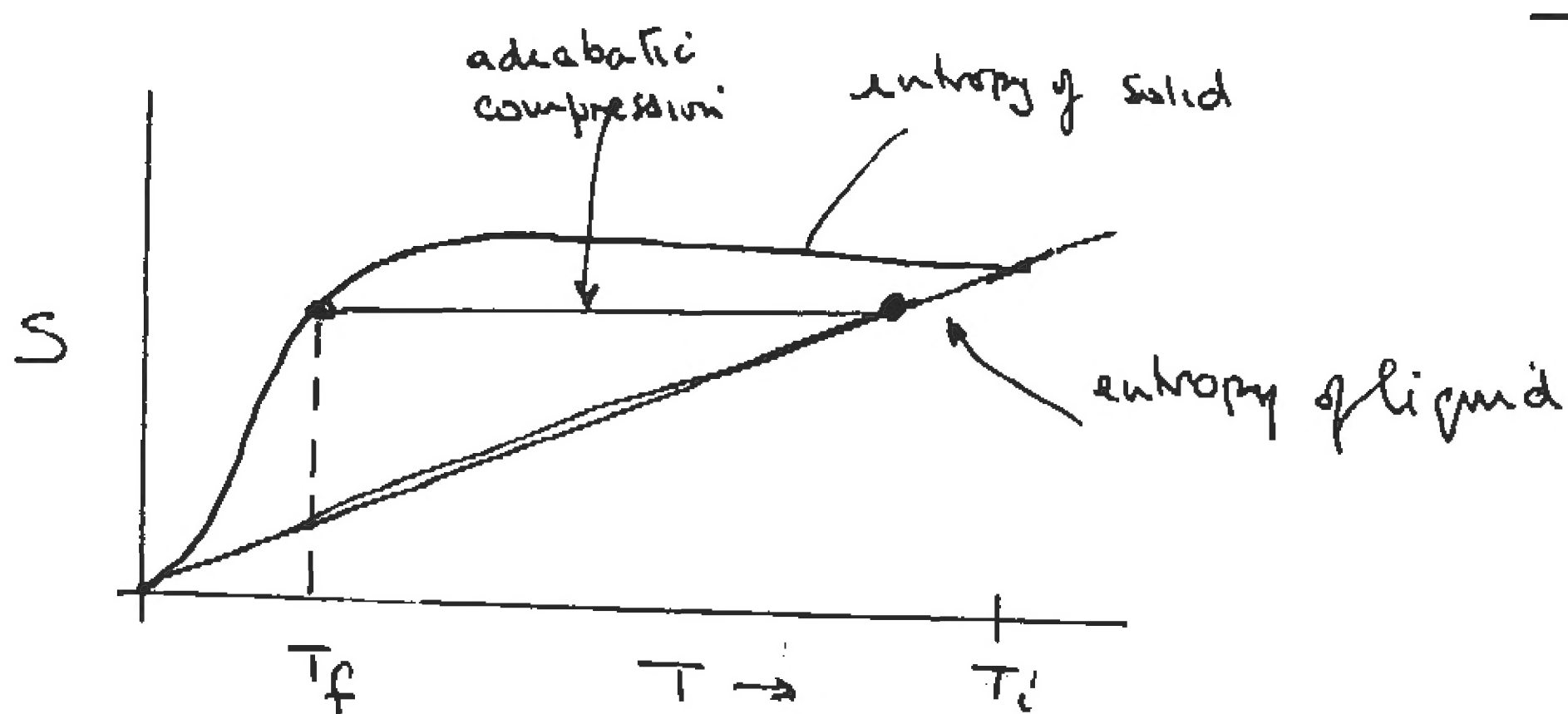
Reason - entropy caused by spin disorder - more disorder in solid (Boltzmann distribution) than in liquid (Fermi-Dirac)

Technique compress liquid \rightarrow solid adiabatically

- reduces temp

$$T_i \sim 0.3K$$

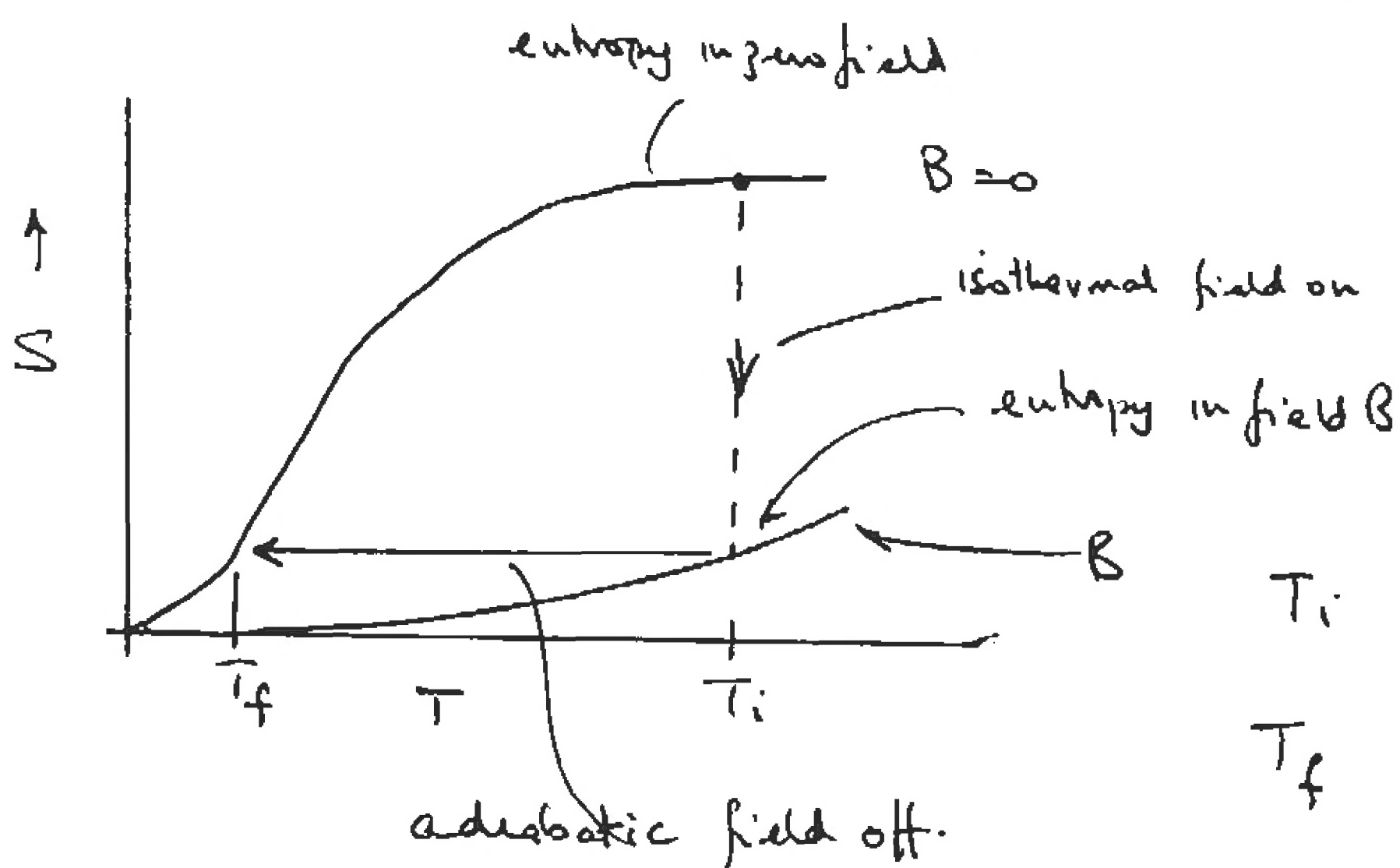
$$T_f \sim 5 \times 10^{-3} K.$$



(iv) Cooling by adiabatic demagnetisation - logical account including some of points.

Solid paramagnetic sample

At start temp T_i - entropy reduced by isothermal application of stray field B . - energy given out - taken away by fridge in contact.



Contact with fridge broken

Field off adiabatically

temp drops $T_i \rightarrow T_f$

atomic

$$T_i \sim 1K$$

nuclear

$$5 \times 10^{-3} K$$

$$T_f \sim 5 \times 10^{-3} K$$

$$5 \times 10^{-6} K$$

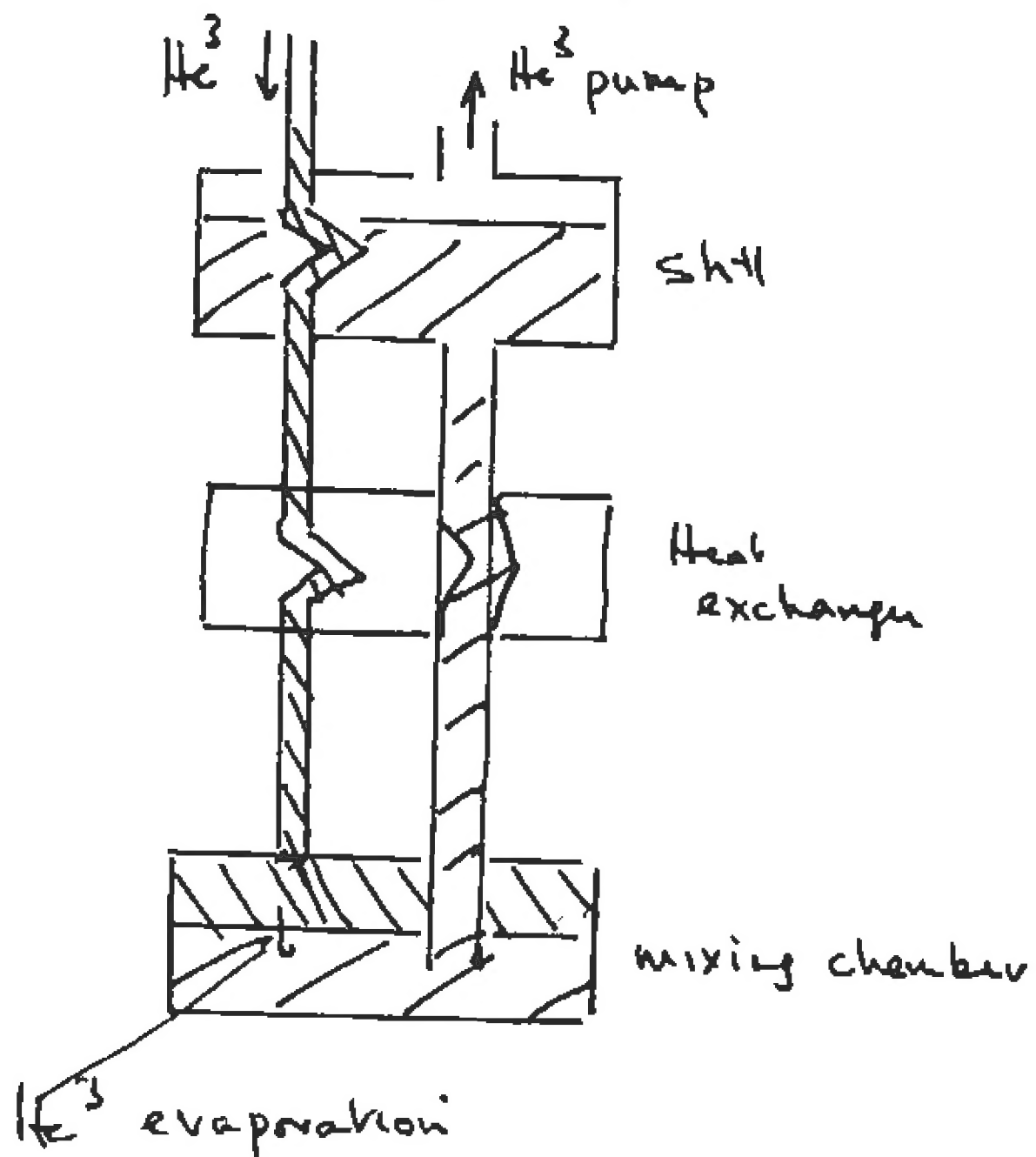
(v) Dilution refrigerator - logical account including some of points

Working substance liquid $\text{He}^3 / \text{He}^4$ mixture

At $T < 0.86 \text{ K}$ mix separates to He^3 rich phase and He^4 rich phase

In mixing chamber evaporation of He^3 from He^3 rich to He^4 rich phase causes 'evaporation cooling'.

Rest of apparatus ensures that this is continuous process.



He^3 is pumped preferentially from He^4 rich phase and recirculated back into the He^3 rich phase to keep 'evaporation' cooling going in mixing chamber.

5